Modularity of trace functions in orbifold theory for $\mathbb{Z}$-graded vertex operator superalgebras

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Abstract

We study the trace functions in orbiford theory for $\mathbb{Z}$-graded vertex operator superalgebras and obtain a modular invariance result. More precisely, let V be a C_2-cofinite $\mathbb{Z}$-graded vertex operator superalgebra and G a finite automorphism group of V. Then for any commuting pairs $(g,h) \in G$, the $h\sigma$-trace functions associated to the simple g-twisted V-modules are holomorphic in the upper half plane where σ is the canonical involution on V coming from the superspace structure of V. If V is further g-rational for every $g \in G$, the trace functions afford a representation for the full modular group $SL(2,\mathbb{Z})$.

1 Introduction

This work is a continuation of our study of the modular invariance for trace functions in orbifold theory. Motivated by the generalized moonshine [N] and orbifold theory in physics [DVVV], the modular invariance of trace functions in orbifold theory has been studied for an arbitrary vertex operator algebra [DLM3]. This work has be generalized to a $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebra [DZ2] (also see [H]). In this paper we investigate the modular invariance of trace functions in orbifold theory for a $\mathbb{Z}$-graded vertex operator superalgebra.

It is true that many $\mathbb{Z}$-graded vertex operator superalgebra can be obtained from a $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebra by changing the Virasoro element (cf. [DM2]). In this case we can apply the results from [DZ1] and [DZ2] to these $\mathbb{Z}$-graded vertex operator superalgebras without extra work. Unfortunately, there are many $\mathbb{Z}$-graded vertex operator superalgebras which can not be obtained in this way. So an independent study of $\mathbb{Z}$-graded vertex operator superalgebra becomes necessary although the main ideas and methods in this paper are similar to those used in [Z], [DLM3] and [DZ2].

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There is a subtle difference among these modular invariance results. In order to explain this we fix a finite automorphism group of the vertex operator (super)algebra. We use g and h for two commuting elements in G. For the vertex operator superalgebra, there is a special automorphism σ of order 2 coming from the structure of the superspace. The involution σ can be expressed as $(-1)^F$ in the physics literature (cf. [GSW], [P]) where F is the fermion number. Here is the difference: for a vertex operator algebra, the space of all h-trace on g-twisted sectors is modular invariant [DLM3], for $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebra, the space of all $h\sigma$-trace on $g\sigma$-twisted sectors is modular invariant [DZ2], and for $\mathbb{Z}$-graded vertex operator superalgebra, the space of all $h\sigma$-trace on g-twisted sectors is modular invariant. It is worthy to point out that the $h\sigma$-trace in the physics literature is called the super trace.

Since the setting and most results in this paper are similar to those in [DLM2], [DLM3], [DZ1], and [DZ2], we refer the reader in many places to these papers for details.

The organization of this paper is as follows: In section 2, we review the definition of $\mathbb{Z}$-graded vertex operator superalgebra (VOSA) and various notions of g-twisted modules. Section 3 is devoted to studying the representation theory for $\mathbb{Z}$-graded VOSA. We introduce the associative algebra $A_g(V)$, and investigate the relation between the g-twisted modules and and $A_g(V)$-modules. Section 4 is the heart of the paper. We give the definition of 1-point functions on the torus and also establish the modular invariance property of it. We prove that for a simple g-twisted module M, g and h are two commuting elements in $Aut(V)$, M is σ, h-stable, then the $h\sigma$-trace functions for M are 1-point function. Moreover, when V is g-rational, the collection of trace functions associated to the collection of inequivalent simple $h\sigma, h$ stable g-twisted V modules form a basis of $\mathcal{C}(g,h)$. In Section 5 we discuss an example to show the modularity of trace functions.

2 $\mathbb{Z}$-graded vertex operator superalgebras

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be $\mathbb{Z}_2$-graded vector space. For any $v \in V_{\bar{i}}$ with $i = 0, 1$ we define $\tilde{v} = i$. Moreover, let $\epsilon_{\mu,v} = (-1)^{\tilde{u}\tilde{v}}$ and $\epsilon_v = (-1)^{\tilde{v}}$.

Definition 2.1 A $\mathbb{Z}$-graded vertex operator superalgebra *($\mathbb{Z}$-graded VOSA) is a $\mathbb{Z} \times \mathbb{Z}_2$-graded vector space*

$$V = \bigoplus_{n \in \mathbb{Z}} V_n = V_{\bar{0}} \oplus V_{\bar{1}} = \bigoplus_{n \in \mathbb{Z}} (V_{\bar{0},n} \oplus V_{\bar{1},n}) \quad (\mathrm{wt} v = n \text{ if } v \in V_n)$$

together with two distinct vectors $\mathbf{1} \in V_{\bar{0},0}$, $\omega \in V_{\bar{0},2}$ *and equipped with a linear map*

$$\begin{aligned} &V \to (\operatorname{End} V)[[z, z^{-1}]], \\ &v \mapsto Y(v, z) = \sum_{n\in\mathbb{Z}} v(n) z^{-n-1} \qquad (v(n) \in \operatorname{End} V) \end{aligned}$$

satisfying the following axioms for $u, v \in V$:

(i) $u(n)v = 0$ *for sufficiently large* n;

(ii) If $u \in V_{\bar{i}}$ *and* $v \in V_{\bar{j}}$, *then* $u(n)v \in V_{\bar{i}+\bar{j}}$ *for all* $n \in \mathbb{Z}$;

(iii) $Y(\mathbf{1}, z) = Id_V$ *and* $Y(v, z)\mathbf{1} = v + \sum_{n\geq 2} v(-n)\mathbf{1}z^{n-1}$;

(iv) Set $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ *then*

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c \tag{2.1}$$

where $c \in \mathbb{C}$ *is called the* central charge, *and*

$$L(0)|_{V_n} = n, \ \ n \in \mathbb{Z}, \tag{2.2}$$

$$\frac{d}{dz}Y(v, z) = Y(L(-1)v, z); \tag{2.3}$$

(v) For $\mathbb{Z}_2$*-homogeneous* $u, v \in V$,

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) - \epsilon_{u,v} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1) \\ &\qquad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2) \end{aligned} \tag{2.4}$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ *and* $(z_i - z_j)^n$ *is expanded in nonnegative powers of* z_j *and* z_0, z_1, z_2, *etc. are independent commuting formal variables.*

Following the proof of Theorem 4.21 of [Z] we can also define $\mathbb{Z}$-graded vertex operator superalgebra on a torus associated to V.

Theorem 2.2 $(V, Y[\,], \mathbf{1}, \tilde{\omega})$ *is a* $\mathbb{Z}$*-graded vertex operator superalgebra, where* $\tilde{\omega} = \omega - \frac{c}{24}$ *and for homogeneous* $v \in V$

$$Y[v, z] = Y(v, e^z - 1)e^{z\operatorname{wt} v} = \sum_{n\in\mathbb{Z}} v[n]z^{-n-1} \tag{2.5}$$

Let $Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} L[n]z^{-n-2}$. Then $V = \bigoplus_{n\in\mathbb{Z}} V_{[n]}$ is again $\mathbb{Z}$-graded and $L[0] = n$ on $V_{[n]}$. We will write $\operatorname{wt}[v] = n$ if $v \in V_{[n]}$.

Definition 2.3 *A linear automorphism* g *of a* $\mathbb{Z}$*-graded VOSA* V *is called an* automorphism *of* V *if* g *preserves* $\mathbf{1}$, ω *and each* $V_{\bar{i}}$, *and*

$$gY(v, z)g^{-1} = Y(gv, z)$$

for $v \in V$.

Note that if V is a $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebra, the assumption that g preserves each $V_{\bar{i}}$ in unnecessary (cf. [DZ2]).

We denote the full automorphism group by $Aut(V)$. If we define an action , say σ on V associated to the superspace structure of V via $\sigma|V_{\bar{i}} = (-1)^i$. Then σ is a central element of $\mathrm{Aut}(V)$ and will play a special role as in [DZ2].

Let g be an automorphism of V of finite order T. Then we have the following eigenspace decomposition:

$$V = \oplus_{r\in\mathbb{Z}/T\mathbb{Z}} V^r \tag{2.6}$$

where $V^r = \{v \in V | gv = e^{-2\pi ir/T}v\}$. We now give various notions of $g-$ twisted V-modules.

Definition 2.4 *A* weak *g-twisted V-module is a $\mathbb{C}$-linear space M equipped with a linear map*

$$\begin{array}{rcc} V & \to & (\mathrm{End}M)[[z^{1/T}, z^{-1/T}]] \\ v & \mapsto & Y_M(v,z) = \sum_{n\in\mathbb{Q}} v(n)z^{-n-1} \end{array}$$

which satisfies:

(i) $v(m)w = 0$ for $v \in V, w \in M$ and $m >> 0$;

(ii) $Y_M(\mathbf{1}, z) = Id_M$;

(iii) For $v \in V^r$ and $0 \le r \le T-1$

$$Y_M(v,z) = \sum_{n\in\frac{r}{T}+\mathbb{Z}} v(n)z^{-n-1};$$

(iv) For $u \in V^r$,

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u,z_1)Y_M(v,z_2) - \epsilon_{u,v} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v,z_2)Y_M(u,z_1) \\ & \quad = z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u,z_0)v,z_2) \end{aligned} \tag{2.7}$$

Set

$$Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}.$$

Then we have $Y_M(L(-1)v,z) = \frac{d}{dz}Y_M(v,z)$ for $v \in V$ and $L(n)$ also satisfy the Virasoro algebra relation with central charge c (see [DLM1]).

Definition 2.5 *A weak g-twisted V-module M is* admissible *if it is $\frac{1}{T}\mathbb{Z}_+$-graded:*

$$M = \bigoplus_{0\le n\in\frac{1}{T}\mathbb{Z}} M(n) \tag{2.8}$$

such that for homogeneous $v \in V$,

$$v(m)M(n) \subseteq M(n + \mathrm{wt}v - m - 1) \tag{2.9}$$

We may and do assume that $M(0) \neq 0$ if $M \neq 0$.

Definition 2.6 *A weak g-twisted V-module M is called (ordinary) g-twisted V-module if it is a $\mathbb{C}$-graded with*

$$M = \coprod_{\lambda \in \mathbb{C}} M_\lambda \tag{2.10}$$

where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$ such that $\dim M_\lambda$ *is finite and for fixed λ, $M_{\frac{n}{T}+\lambda} = 0$ for all small enough integers n.*

It is not hard to prove that any ordinary g-twisted V-module is admissible. The notion of weak, admissible and ordinary V-modules is just the special case when $g = 1$.

If M is a simple g-twisted V-module, then

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda + n/T} \tag{2.11}$$

for some $\lambda \in \mathbb{C}$ such that $M_\lambda \neq 0$ (cf. [Z]). λ is defined to be the *conformal weight* of M.

Definition 2.7 *(i) A $\mathbb{Z}$-graded VOSA V is called g-rational for an automorphism g of finite order if the category of admissible modules is completely reducible. V is called rational if it's 1-rational.*

(ii) V is called holomorphic if V is rational and V is the only irreducible V-module up to isomorphism.

(iii) V is called g-regular if any weak g-twisted V-module is a direct sum of irreducible ordinary g-twisted V-modules.

As in [DLM2] and [DZ1] we have the following result.

Theorem 2.8 *If V is g-rational $\mathbb{Z}$-graded VOSA with $g \in Aut(V)$ being of finite order then*

(i) There are only finitely many irreducible admissible g-twisted V-modules up to isomorphism.

(ii) Each irreducible admissible g-twisted V-module is ordinary.

3 The associative algebra $A_g(V)$

In this section we construct the associative algebra $A_g(V)$ and study the relation between admissible g-twisted V-modules and $A_g(V)$-modules. The result is similar to those obtained in [DLM2] (also see [Z], [KW], [X], [DZ1]).

As before we assume that the order of g is T. For $0 \le r \le T-1$ we define $\delta_r = \delta_{r,0}$. Let $O_g(V)$ be the linear span of all $u \circ_g v$, where for homogeneous $u \in V^r$(cf. (2.6)) and $v \in V$,

$$u \circ_g v = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt} u - 1 + \delta_r + \frac{r}{T}}}{z^{1+\delta_r}} Y(u,z)v. \tag{3.1}$$

Set $A_g(V) = V/O_g(V)$ and define a second linear product $*_g$ on V for the above u, v as follows:

$$u *_g v = \mathrm{Res}_z Y(u,z) \frac{(1+z)^{\mathrm{wt}\, u}}{z} v \tag{3.2}$$

if $r = 0$ and $u *_g v = 0$ if $r > 0$. It is easy to see that $A_g(V)$ is in fact a quotient of V^0.

As in [DLM2],[X] and [DZ2] we have

Theorem 3.1 *$A_g(V) = V/O_g(V)$ is an associative algebra with identity $\mathbf{1} + O_g(V)$ under the product $*_g$. Moreover, $\omega + O_g(V)$ lies in the center of $A_g(V)$.*

For a weak g-twisted V-module M, we define the space of the lowest weight vectors

$$\Omega(M) = \{w \in M | u(\mathrm{wt} u - 1 + n)w = 0, u \in V, n > 0\}.$$

We have (see [DLM2]):

Theorem 3.2 *Let M be a weak g-twisted V-module. Then*

(i) $\Omega(M)$ is an $A_g(V)$-module such that $v + O_g(V)$ acts as $o(v)$.

(ii) If $M = \sum_{n \ge 0} M(n/T)$ is an admissible g-twisted V-module such that $M(0) \neq 0$, then $M(0) \subset \Omega(M)$ is an $A_g(V)$-submodule. Moreover, M is irreducible if and only if $M(0) = \Omega(M)$ and $M(0)$ is a simple $A_g(V)$-module.

(iii) The map $M \to M(0)$ gives a 1-1 correspondence between the irreducible admissible g-twisted V-modules and simple $A_g(V)$-modules.

We also have (see [DLM2]):

Theorem 3.3 *Suppose that V is a g-rational vertex operator superalgebra. Then the following hold:*

(i) $A_g(V)$ is a finite dimensional semisimple associative algebra.

(ii) V has only finitely many irreducible admissible g-twisted modules up to isomorphism.

(iii) Every irreducible admissible g-twisted V-module is ordinary.

(iv) V is g^{-1}-rational.

4 Modularity of trace functions

We are working in the setting of section 5 in [DLM3]. In particular, g, h are commuting elements in $Aut(V)$ with finite orders $o(g) = T, o(h) = T_1$, A is the subgroup of $Aut(V)$ generated by g and h, $N = lcm(T, T_1)$ is the exponent of A, $\Gamma(T, T_1)$ is the subgroup of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, \mathbb{Z})$ satisfying $a \equiv d \equiv 1 \pmod N$, $b \equiv 0 (\mathrm{mod}\ T)$, $c \equiv 0 (\mathrm{mod}\ T_1)$ and $M(T, T_1)$ be the ring of holomorphic modular forms on $\Gamma(T, T_1)$ with natural gradation $M(T, T_1) = \oplus_{k \geq 0} M_k(T, T_1)$, where $M_k(T, T_1)$ is the space of forms of weight k. Then $M(T, T_1)$ is a Noetherian ring.

Recall the Bernoulli polynomials $B_r(x) \in \mathbb{Q}[x]$ defined by

$$\frac{te^{tx}}{(e^t - 1)} = \sum_{r=0}^{\infty} \frac{B_r(x) t^r}{r!}.$$

For even $k \geq 2$, the normalized Eisenstein series $E_k(\alpha)$ is given by

$$E_k(\tau) = \frac{-B_k}{k!} + \frac{2}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{4.1}$$

Also introduce

$$\begin{aligned} Q_k(\mu, \lambda, q_\tau) &= Q_k(\mu, \lambda, \tau) \\ &= \frac{1}{(k-1)!} \sum_{n \geq 0} \frac{\lambda (n + j/T)^{k-1} q_\tau^{n+j/T}}{1 - \lambda q_\tau^{n+j/T}} \\ &\quad + \frac{(-1)^k}{(k-1)!} \sum_{n \geq 1} \frac{\lambda^{-1} (n - j/T)^{k-1} q_\tau^{n-j/T}}{1 - \lambda^{-1} q_\tau^{n-j/T}} - \frac{B_k(j/T)}{k!} \end{aligned} \tag{4.2}$$

for $(\mu, \lambda) = (e^{\frac{2\pi i j}{T}}, e^{\frac{2\pi i l}{T_1}})$ and $(\mu, \lambda) \neq (1, 1)$,when $k \geq 1$ and $k \in \mathbb{Z}$. Here $(n + j/T)^{k-1} = 1$ if $n = 0, j = 0$ and $k = 1$. Similarly, $(n - j/T)^{k-1} = 1$ if $n = 1, j = M$ and $k = 1$. We also define

$$Q_0(\mu, \lambda, \tau) = -1. \tag{4.3}$$

It is proved in [DLM3] that E_{2k}, Q_r are contained in $M(T, T_1)$ for $k \geq 2$ and $r \geq 0$.

Set $V(T, T_1) = M(T, T_1) \otimes_{\mathbb{C}} V.$. Given $v \in V$ with $gv = \mu^{-1} v, hv = \lambda^{-1} v$ we define a vector space $O(g, h)$ which is a $M(T, T_1)$-submodule of $V(T, T_1)$ consisting of the following elements:

$$v[0]w, w \in V, (\mu, \lambda) = (1, 1) \tag{4.4}$$

$$v[-2]w + \sum_{k=2}^{\infty}(2k-1)E_{2k}(\tau) \otimes v[2k-2]w, (\mu, \lambda) = (1,1) \tag{4.5}$$

$$v, (\epsilon_v, \mu, \lambda) \neq (1,1,1) \tag{4.6}$$

$$\sum_{k=0}^{\infty} Q_k(\mu, \lambda, \tau) \otimes v[k-1]w, (\mu, \lambda) \neq (1,1). \tag{4.7}$$

Definition 4.1 *Let $\mathfrak{h}$ denote the upper half plane. The space of (g,h) 1-point functions $\mathcal{C}(g,h)$ is defined to be the $\mathbb{C}$-linear space consisting of functions*

$$S : V(T, T_1) \times \mathfrak{h} \to \mathbb{C}$$

s.t

(i) $S(v,\tau)$ is holomorphic in τ for $v \in V(T,T_1)$.
(ii) $S(v,\tau)$ is $\mathbb{C}$ linear in v and for $f \in M(T,T_1)$, $v \in V$,

$$S(f \otimes v, \tau) = f(\tau)S(v,\tau)$$

(iii) $S(v,\tau) = 0$ if $v \in O(g,h)$.
(iv) If $v \in V$ with $\sigma v = gv = hv = v$, then

$$S(L[-2]v, \tau) = \partial S(v,\tau) + \sum_{l=2}^{\infty} E_{2l}(\tau) S(L[2l-2]v, \tau). \tag{4.8}$$

Here ∂S is the operator which is linear in v and satisfies

$$\partial S(v,\tau) = \partial_k S(v,\tau) = \frac{1}{2\pi i}\frac{d}{d\tau} S(v,\tau) + kE_2(\tau)S(v,\tau) \tag{4.9}$$

for $v \in V_{[k]}$.

We have the following modular invariance result (see Theorem 5.4 of [DLM3]):

Theorem 4.2 *For $S \in \mathcal{C}(g,h)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we define*

$$S|\gamma(v,\tau) = S|_k\gamma(v,\tau) = (c\tau + d)^{-k} S(v, \gamma\tau) \tag{4.10}$$

for $v \in V_{[k]}$, and extend linearly. Then $S|\gamma \in \mathcal{C}((g,h)\gamma)$.

Let g, h, σ, V be as before, and M be a simple g-twisted module. We now show how the graded $h\sigma$-trace functions on g-twisted V-modules produce (g,h) 1-point functions.

From (2.11), we know that if M is a simple $g-$twisted module then there exists a complex number λ such that

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda + \frac{n}{T}} \tag{4.11}$$

Now we define a $(h\sigma)g(h\sigma)^{-1}$-twisted V-module $(h\sigma\circ M, Y_{h\sigma\circ M})$ such that $h\sigma\circ M = M$ as vector spaces and

$$Y_{h\sigma\circ M}(v,z) = Y_M((h\sigma)^{-1}v,z).$$

Since g, h, σ commute each other, $h\sigma\circ M$ is, in fact, a simple g-twisted V-module again. The M is called *h-stable* if $h\sigma\circ M$ and M are isomorphic g-twisted V-modules. In this case, there is a linear map $\phi(h\sigma): M\to M$ such that

$$\phi(h\sigma)Y_M(v,z)\phi(h\sigma)^{-1} = Y_M((h\sigma)v,z) \tag{4.12}$$

for all $v\in V$.

We now assume that M is h-stable. For homogeneous $v\in V$, we define the trace function T as follows:

$$T(v) = T_M(v,(g,h),q) = z^{\mathrm{wt}v}\mathrm{tr}_M Y_M(v,z)\phi(h\sigma)q^{L(0)-\frac{c}{24}} \tag{4.13}$$

Here c is the central charge of V. Note that for $m\in\frac{1}{T}\mathbb{Z}, v(m)$ maps M_μ to $M_{\mu+\mathrm{wt}v-m-1}$. Hence

$$T(v) = q^{\lambda-\frac{c}{24}}\sum_{n=0}^{\infty}\mathrm{tr}_{M_{\lambda+\frac{n}{T}}}o(v)\phi(h\sigma)q^{\frac{n}{T}} = \mathrm{tr}_M o(v)\phi(h\sigma)q^{L(0)-\frac{c}{24}}. \tag{4.14}$$

In order to state the next theorem we need to recall C_2-cofinite condition from [Z]. V is called C_2-cofinite if $V/C_2(V)$ is finite dimensional where $C_2(V) = \{u_{-2}v|u,v\in V\}$.

Theorem 4.3 *Suppose that V is C_2-cofinite, $g,h\in\mathrm{Aut}(V)$commute and have finite orders. Let M be a simple g-twisted V-module such that M is h and σ-stable. Then the trace function $T_M(v,(g,h),q)$ converges to a holomorphic function in the upper half plane $\mathfrak{h}$ where $q=e^{2\pi i\tau}$ and $\tau\in\mathfrak{h}$. Moreover, $T_M\in\mathcal{C}(g,h)$.*

The proof of this theorem is similar to Theorem 4.3 of [DZ2] although the idea goes back to [Z] and [DLM3].

We also have the following theorems.

Theorem 4.4 *Let $M^1, M^2, ...M^s$ be the collection of inequivalent simple $h\sigma$ and σ-stable g-twisted V-modules, then the corresponding trace functions T_1, T_2,...T_s (4.13) are independent vectors of $\mathcal{C}(g,h)$. Moreover, if V is g-rational, $T_1, T_2, ...T_s$ form a basis of $\mathcal{C}(g,h)$.*

The following theorem is an immediate consequence of Theorem 4.3 and Theorem4.4.

Theorem 4.5 *Suppose that V is a C_2-cofinite vertex operator superalgebra and G a finite group of automorphisms of V. Assume that V is x-rational for each $x \in \bar{G}$. Let $v \in V$ satisfy $\mathrm{wt}[v] = k$. Then the space of (holomorphic) functions in $\mathfrak{h}$ spanned by the trace functions $T_M(v,(g,h),\tau)$ for all choices of g,h in G and σ,h-stable M is a (finite-dimensional) $SL(2,\mathbb{Z})$-module such that*

$$T_M|\gamma(v,(g,h),\tau) = (c\tau+d)^{-k}T_M(v,(g,h),\gamma\tau),$$

where $\gamma \in SL(2,\mathbb{Z})$ acts on $\mathfrak{h}$ as usual.

More precisely, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ then we have an equality

$$T_M(v,(g,h),\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k \sum_W \gamma_{M,W} T_W(v,(g^a h^c, g^b h^d),\tau),$$

where W ranges over the $g^a h^c$-twisted sectors which are $g^b h^d$ and σ-stable. The constants $\gamma_{M,W}$ depend only on M, W and γ only.

Theorem 4.6 *Let V be a rational and C_2-cofinite $\mathbb{Z}$-graded VOSA. Let M^1, M^2,... M^s be the collection of inequivalent simple σ-stable V-modules. Then the space spanned by*

$$T_i(v,\tau) = T_i(v,(1,1),\tau) = tr_{M^i} o(v)\phi(\sigma) q^{L(0)-\frac{c}{24}} \tag{4.15}$$

gives a representation of the modular group. To be more precisely, for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ there exists a $s \times s$ invertible complex matrice (γ_{ij}) such that

$$T_i(v,\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^n \sum_{j=1}^{s} \gamma_{ij} T_j(v,\tau)$$

for all $v \in V_{[n]}$. Moreover, the matrix (γ_{ij}) is independent of v.

Remark 4.7 *It is interesting to notice that the modular invariance result in Theorem 4.6 is different from that for the vertex operator algebras in [Z] and for the $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebras in [DZ2]. In this case of vertex operator algebras, the space of of the graded trace of simple modules is modular invariant [Z]. But for the $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebras, the space of the graded σ trace on the simple σ-twisted modules is modular invariant. In the present situation, the space of the graded σ trace on the simple V-modules is modular invariant.*

One can also obtain the results such as the number of inequivalent , h,σ-stable simple g-twisted V-modules and rationality of central charges and conformal weights for rational vertex operator superalgebras as in [DLM3] and [DZ2].

5 An example

In this section we consider $\mathbb{Z}$-graded VOSA $V_{\mathbb{Z}\alpha}$ and its σ-twisted module $V_{\mathbb{Z}\alpha+\frac{1}{2}\alpha}$ to demonstrate the modular invariance directly.

We are working in the setting of Chapter 8 of [FLM2]. Let $L = \mathbb{Z}\alpha$ be a nondegenerate lattice of rank 1 with $\mathbb{Z}$-valued symmetric $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle$ s.t. $\langle \alpha, \alpha \rangle = 1$. Set $M(1) = \mathbb{C}[\alpha(-n)|n > 0]$ and let $\mathbb{C}[L]$ be the group algebra of the abelian group L. Set $\mathbf{1} = 1 \otimes e^0 \in V_L$ and $\omega = \frac{1}{2}\alpha(-1)\alpha(-1)$.

Recall that a vertex operator (super)algebra is called *holomorphic* if it is rational and the only irreducible module is itself. We have the following theorem (see [B], [FLM2], [D], [DLM1], [DM1]).

Theorem 5.1 *(i) $(V_L, Y, \mathbf{1}, \omega)$ is a holomorphic $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebra with central charge $c = \mathrm{rank}(L) = 1$.*

(ii) $(V_L)_{\bar{0}} = M(1) \otimes \mathbb{C}[2L]$ and $(V_L)_{\bar{1}} = M(1) \otimes \mathbb{C}[2L + \alpha]$.

(iii) $V_{L+\frac{1}{2}\alpha}$ is the unique irreducible σ-twisted module for V_L.

One can verify the next theorem easily.

Theorem 5.2 *(i) If we let $\omega' = \frac{1}{2}\alpha(-1)^2 \pm \frac{1}{2}\alpha(-2)$, then $(V_L, Y, \mathbf{1}, \omega')$ is a holomorphic $\mathbb{Z}$-graded vertex operator superalgebra with central charge $c' = -2$.*

(ii) $V_{L+\frac{1}{2}\alpha}$ is the unique irreducible σ-twisted module for $\mathbb{Z}$-graded vertex operator superalgebra V_L.

We consider the group G to be the cyclic group generated by σ. It is straightforward to compute the following trace functions:

$$\begin{aligned}
T(\mathbf{1}, (1,1), \tau) &= tr_{V_{\mathbb{Z}\alpha}} \sigma q^{L(0)' - \frac{-2}{24}} \\
&= q^{\frac{1}{12}} \sum_{n=0}^{\infty} P(n) q^n \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{s(s-1)}{2}} \\
&= \eta(\tau)^{-1} \theta_1(q),
\end{aligned}$$

$$\begin{aligned}
T(\mathbf{1}, (1,\sigma), \tau) &= tr_{V_{\mathbb{Z}\alpha}} q^{L'(0) - \frac{-2}{24}} \\
&= q^{\frac{1}{12}} \sum_{n=0}^{\infty} P(n) q^n \sum_{s=-\infty}^{\infty} q^{\frac{s(s-1)}{2}} \\
&= \eta(\tau)^{-1} \theta_2(q),
\end{aligned}$$

$$\begin{aligned}
T(\mathbf{1}, (\sigma,\sigma), \tau) &= tr_{V_{\mathbb{Z}\alpha+\frac{1}{2}\alpha}} q^{L'(0) - \frac{-2}{24}} \\
&= q^{\frac{1}{12}} \sum_{n=0}^{\infty} P(n) q^n \sum_{s=-\infty}^{\infty} q^{\frac{(s+\frac{1}{2})(s-\frac{1}{2})}{2}} \\
&= \eta(\tau)^{-1} \theta_3(q),
\end{aligned}$$

$$
\begin{aligned}
T(\mathbf{1},(\sigma,1),\tau) &= tr_{V_{\mathbb{Z}\alpha+\frac{1}{2}\alpha}}\sigma q^{L'(0)-\frac{-2}{24}} \\
&= q^{\frac{1}{12}}\prod_{n=1}^{\infty}(1+q^n)\sum_{s=-\infty}^{\infty}(-1)^s q^{\frac{(s+\frac{1}{2})(s-\frac{1}{2})}{2}} \\
&= \eta(\tau)^{-1}\theta_4(q),
\end{aligned}
$$

where

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n)$$

$$\theta_1(q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} = 0$$

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2}$$

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2}$$

$$\theta_4(q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n^2}.$$

Recall the transformation law for η functions

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau), \quad \eta(-\frac{1}{\tau}) = (-i\tau)^{\frac{1}{2}}\eta(\tau)$$

$$\eta(\frac{\tau+1}{2}) = \frac{\eta(\tau)^3}{\eta(\frac{\tau}{2})\eta(2\tau)}$$

and relations

$$\theta_2(q) = 2\frac{\eta(2\tau)^2}{\eta(\tau)}$$

$$\theta_3(q) = \frac{\eta(\tau)^5}{\eta(2\tau)^2\eta(\frac{\tau}{2})^2}$$

$$\theta_4(q) = \frac{\eta(\frac{\tau}{2})^2}{\eta(\tau)}.$$

The modular transformation property for $T(\mathbf{1},(g,h),\tau)$ for $g,h\in G$ can easily be verified and the result, of course, is the same as what Theorem 4.5 claimed. One can also compute the trace functions for the $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebra V_L notice that the sets of trace functions in two cases are exactly the same. Since V_L and $V(H,\mathbb{Z}+\frac{1}{2})$ with $l=2$ are isomorphic $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebra (the boson-fermion correspondence) one can use the modular invariance result for $V(H,\mathbb{Z}+\frac{1}{2})$ obtained in [DZ2] to check the modular transformation property of the trace functions for the $\mathbb{Z}$-graded vertex operator superalgebra V_L.

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